Tree Graphs and Orthogonal Spanning Tree Decompositions

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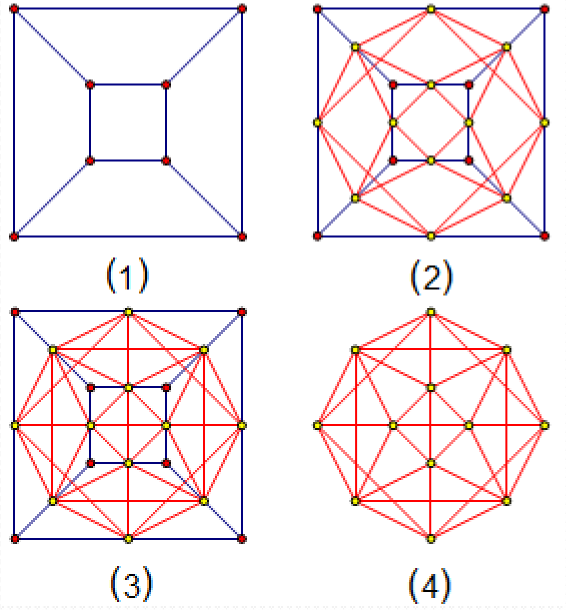
Overview

1. Introduction
2. Background
3. The Tree Graph Function and Parameters
4. Properties of Tree Graphs
5. Trees and Matchings in Complete Graphs

Introduction

- Tree graphs first introduced by Cummins in 1966
- ~20 major papers published since then
- No one has systematically constructed them before
- My two years of research builds on data from dozens of examples



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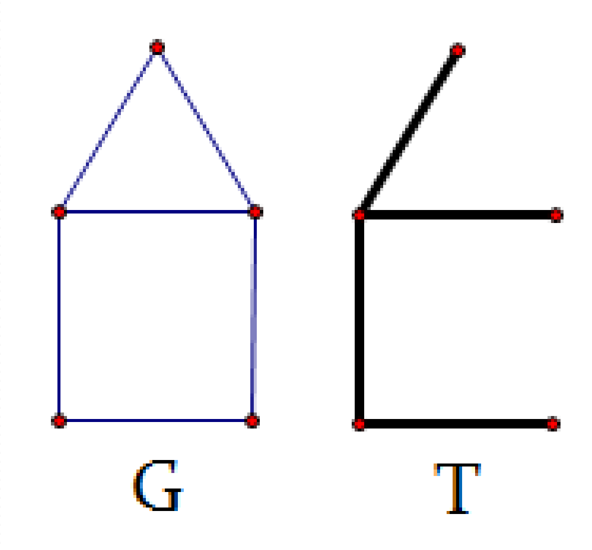
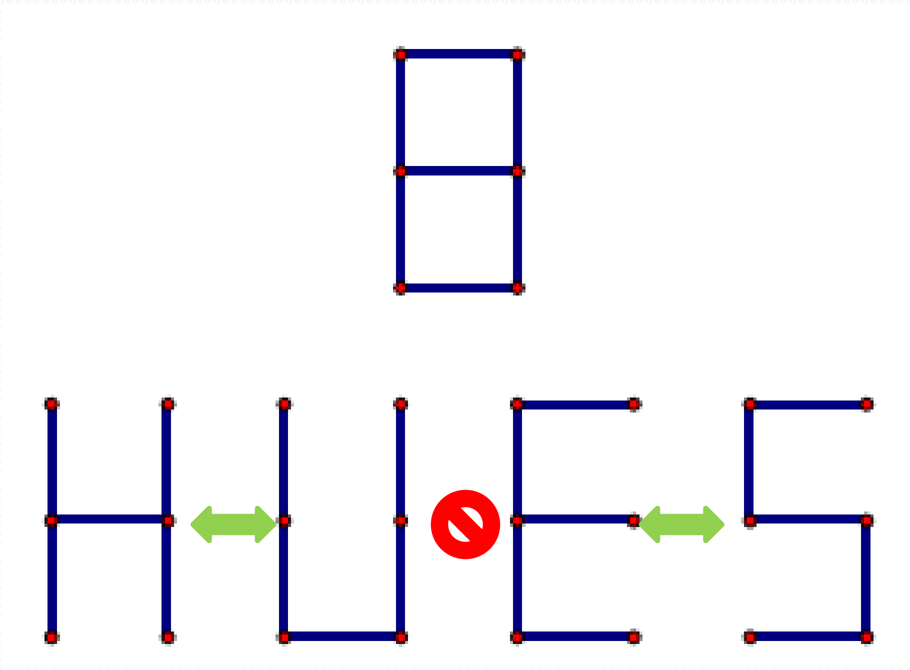
Graphs and Spanning Trees

- Graphs have *vertices* and *edges*
- *Trees* are connected graphs with no cycles
- *Spanning trees* have the same vertices as the original graph
- If a graph has n vertices then a spanning tree will have $n - 1$ edges

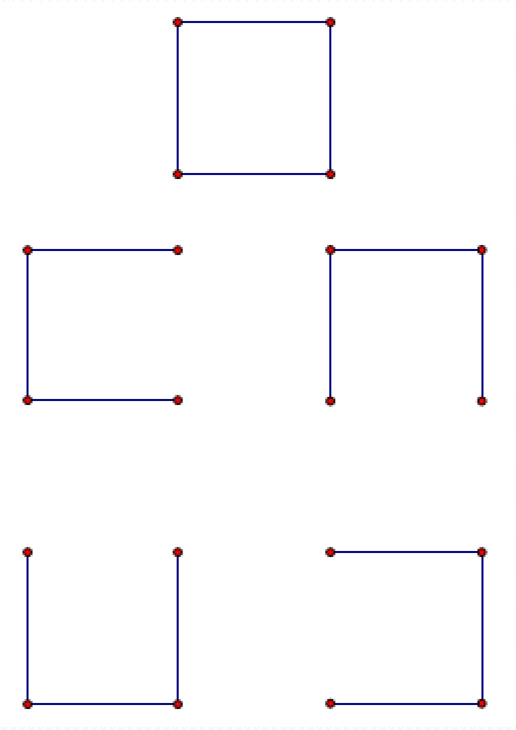


Tree Graphs

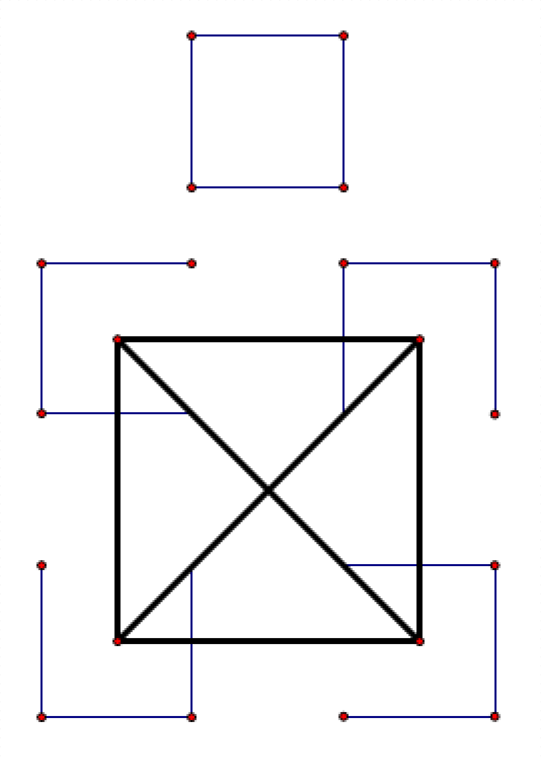
- Let G be a graph. The *tree graph* of G, $T(G)$, has vertices which are the spanning trees of G, where two vertices are adjacent if and only if you can change from one to the other by moving exactly one edge.



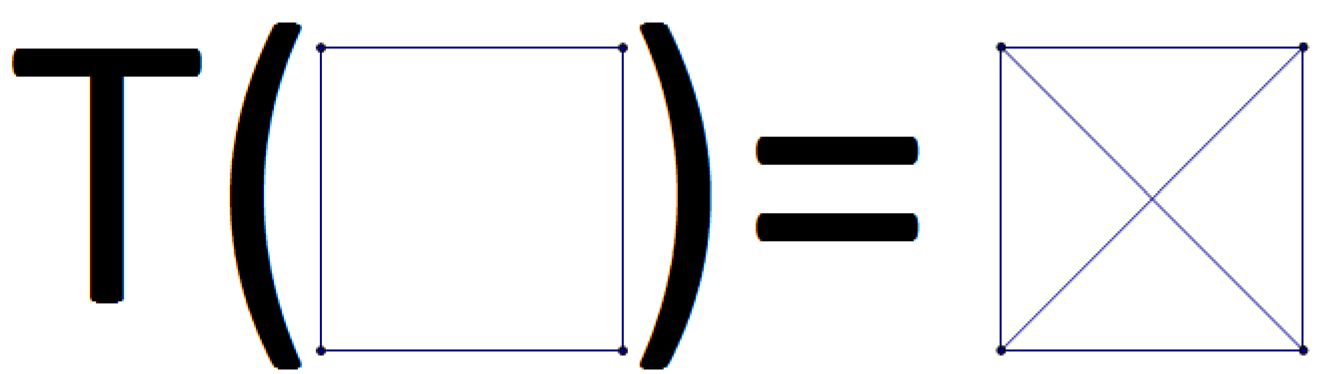
Example: C_4



Example: C_4



Example: C_4



$$T(C_4) = K_4$$

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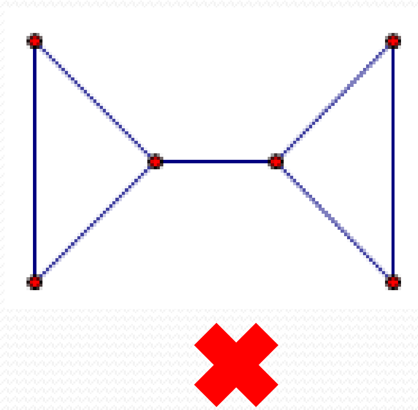
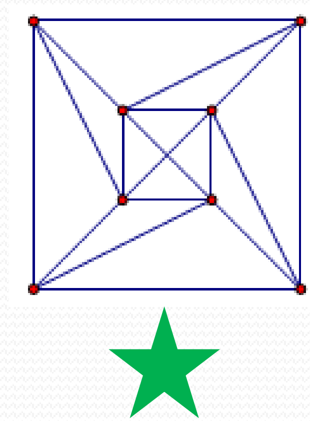
Tree Graph Function & Parameters

- **Thm** (Liu, 1992):

$$\kappa\big(T(G)\big) = \kappa'\big(T(G)\big) = \delta(T(G))$$

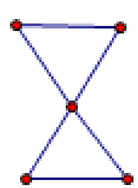
  - Tree graphs are as connected as possible - hard to break apart by removing vertices or edges

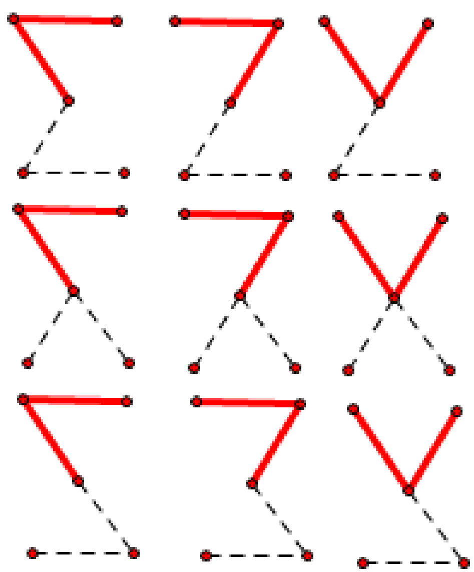


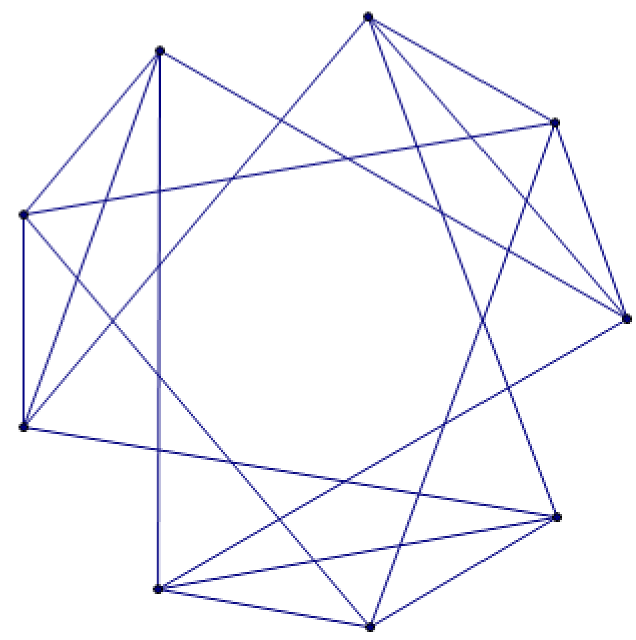


Graphs with Cut Vertices

- Let G and H be graphs and let $G \odot H$ be a graph that joins a vertex in G with a vertex in H.
- **Thm**: $T(G \odot H) \cong T(G) \square T(H)$.
  - Tree graphs of joined graphs are the product of the tree graphs of the pieces



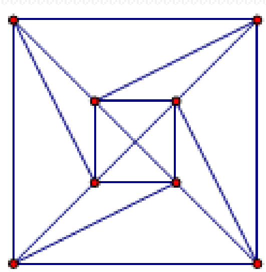




Realizing Tree Graphs

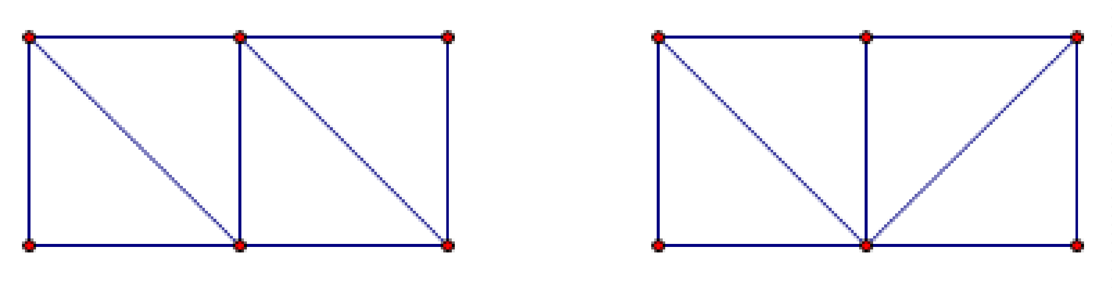
- Given $T(G)$, can we find a graph H such that $T(H) \cong T(G)$?
- What is the pre-image of a tree graph?

Where do I come from?



Isomorphic Tree Graphs

- These pairs of graphs are not isomorphic, but their tree graphs are.
- The starting graphs are *isoparic*: they have the same number of vertices and same number of edges but are not isomorphic.



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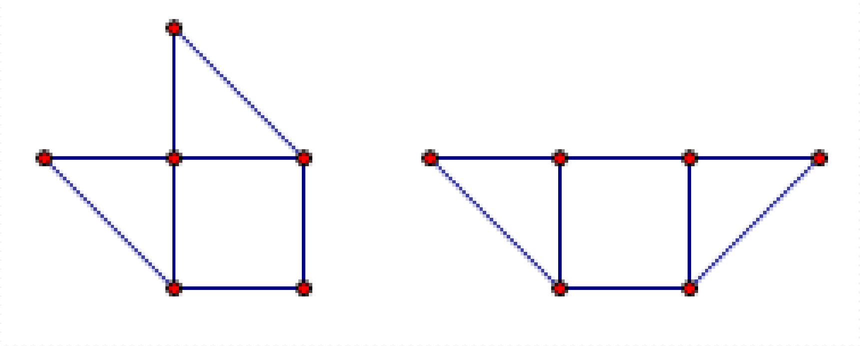
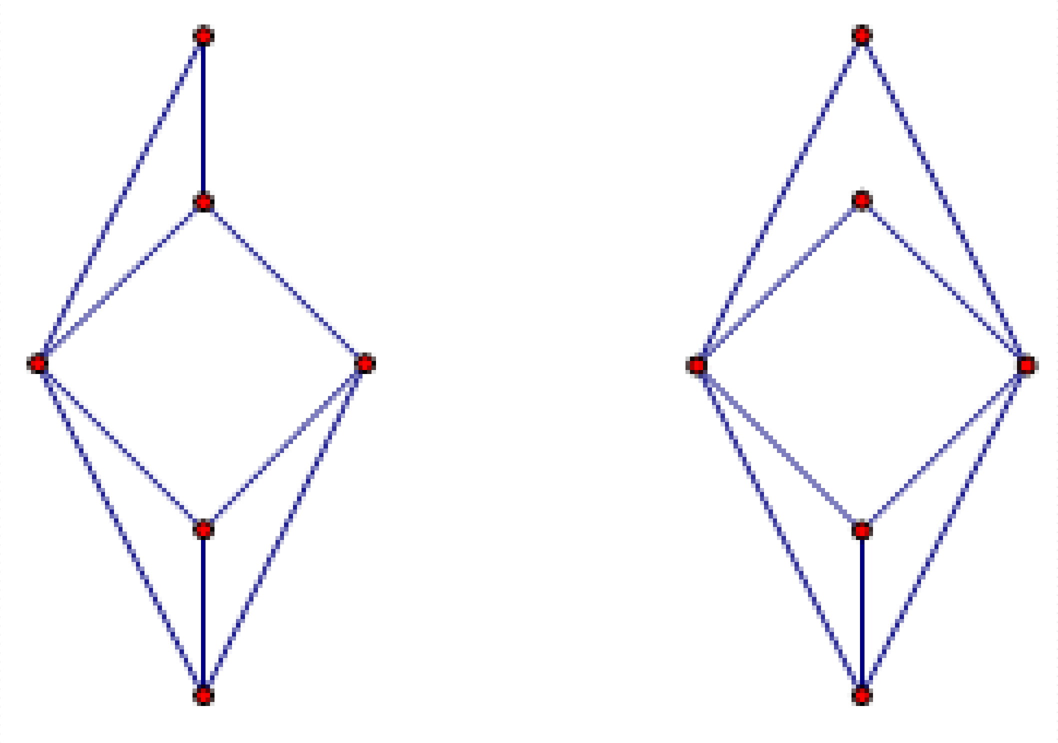


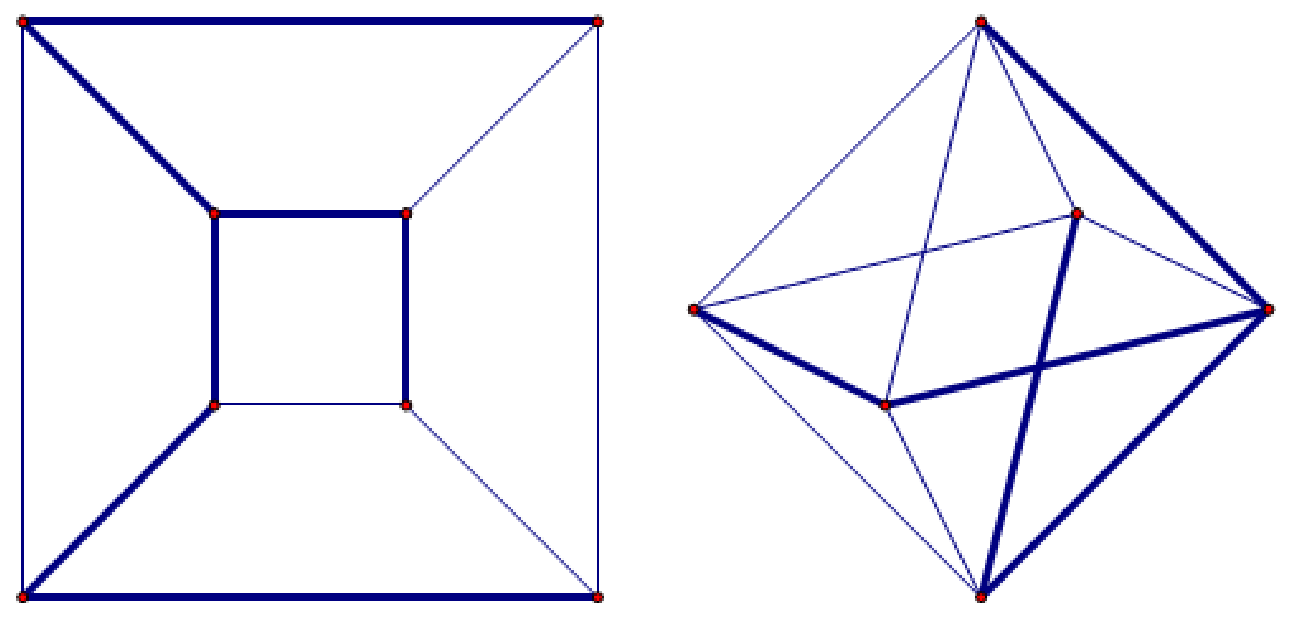
Realizing Tree Graphs

- These two graphs are isoparic and their tree graphs are isoparic (both have 64 vertices and 368 edges).

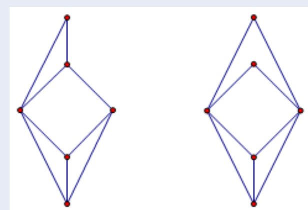
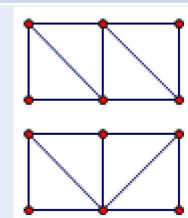
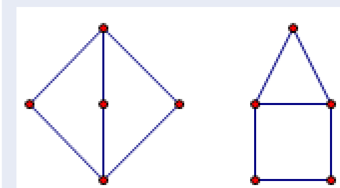
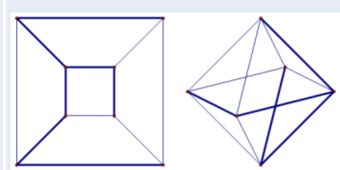


Isomorphic Tree Graphs

- Is it ever the case that $G \ncong H$ but $T(G) \cong T(H)$?
- **Thm**: If G is 3-connected and planar, $T(G) \cong T(G^*)$. Planar duals give isomorphic tree graphs.



Tree Graph Function

		Tree Graphs		
		Isoparic	**Isomorphic**	**Neither**
Starting Graphs	**Isoparic**			
	Isomorphic	Never	Always	Never
	Neither	?	 Non planar duals?	Default

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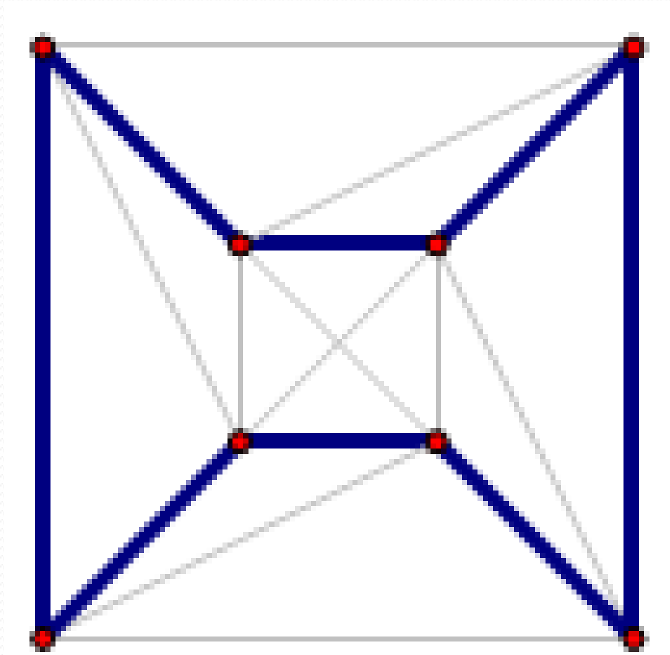
Properties of Tree Graphs

- **Thm** (Cummins, 1966):

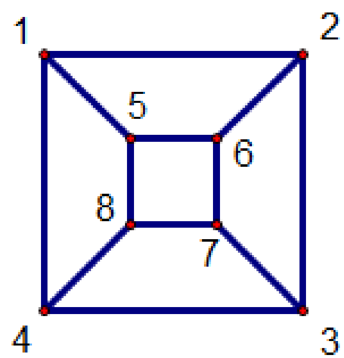
  $T(G)$ is hamiltonian for any graph G

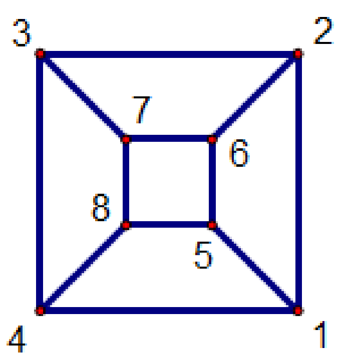
  - There is a cycle through all of the vertices



Symmetry of Tree Graphs

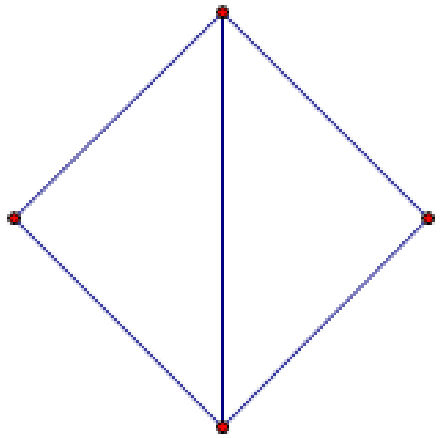
- An *automorphism* of a graph G is a permutation of the vertices that respects adjacency. The set of all automorphisms of G forms a group under composition, $Aut(G)$.
- The *glory* of a graph G, $g(G)$, is the size of its automorphism group. $g(G) = |Aut(G)|$.
- $g(G)$ has been large for most of the small graphs studied so far.

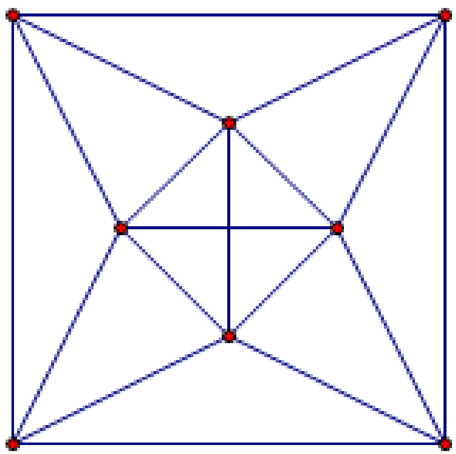




$Aut(T(G))$

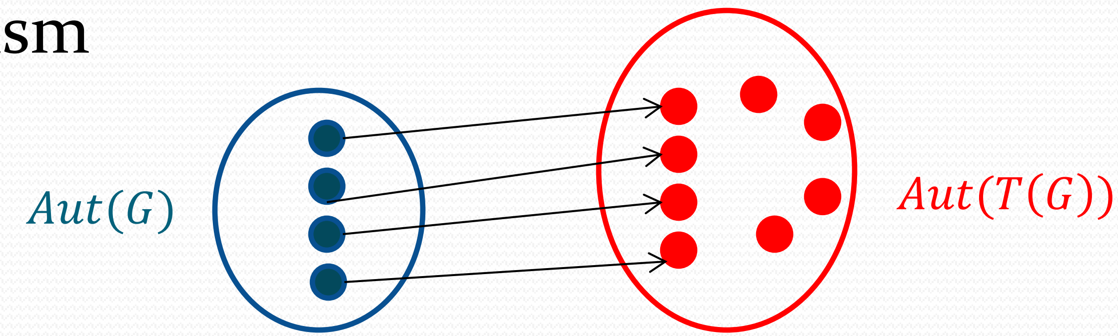
- **Thm**: $Aut(G)$ is a subgroup of $Aut(T(G))$.
  - The symmetries of the input are mirrored in the symmetries of the output.
  - Example: $Aut(K_4 - e) \cong V_4$ while $Aut\big(T(K_4 - e)\big) \cong D_8$, the symmetries of the square.



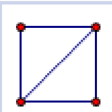
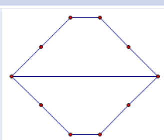
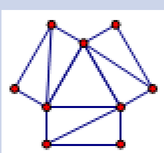
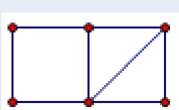


Summary of Proof

- Every graph automorphism σ of G induces a tree graph automorphism ϕ_σ of $T(G)$
- If ϕ_σ fixes all vertices of $T(G)$, then σ fixes all cycle edges of G
- In a 2-connected graph, all edges are cycle edges
- If all edges of G are fixed by σ, all vertices are fixed also
- Therefore map that takes σ to ϕ_σ is an injective homomorphism



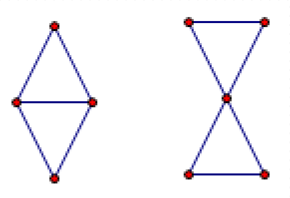
Automorphism Size Examples

Graph G	$g(T(G))$	$g(G)$	Notes
	8	4	D_8 and V_4
$K_{3,2}$	48	12	$S_4 \times S_2$ and $S_3 \times S_2$
K_5	120	120	S_5 and S_5
	28800	4	? and V_4
	288	3	? and $\mathbb{Z}_3$
	12	1	D_{12} and trivial
C_4	24	8	S_4 and D_8

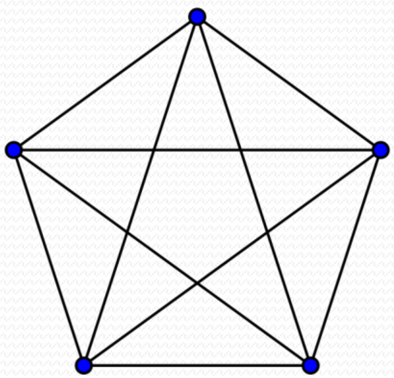
Planarity

- **Thm**: The tree graphs of the diamond and the butterfly are nonplanar. (Contain K_5 and $K_{3,3}$ minors, respectively.)
- **Thm**: $T(G)$ is nonplanar unless $G \cong C_3, C_4$.
  - Cannot draw them flat without lines crossing.

Diamond

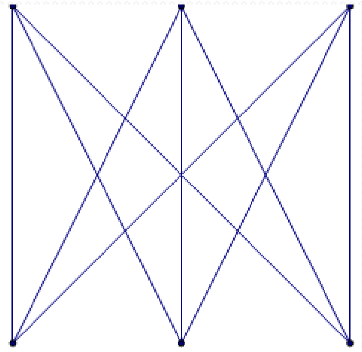


Butterfly



$$T(H) \dot{\leq} T(G)$$

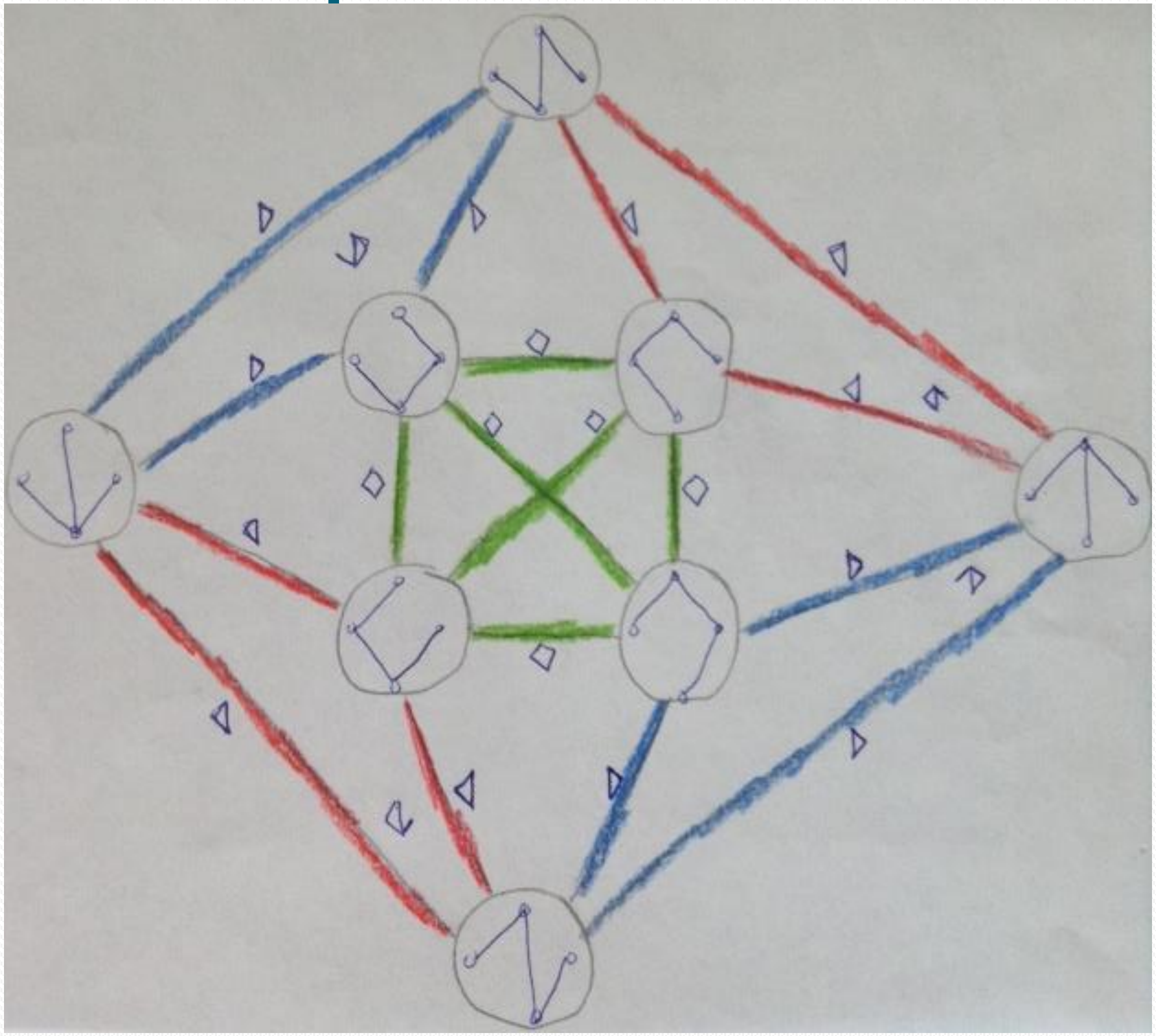
$$H \sqsubseteq G$$

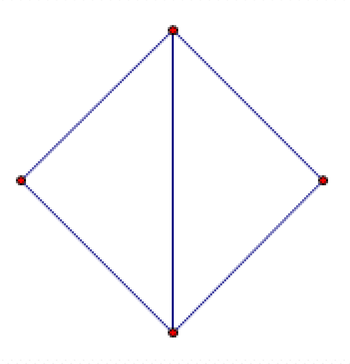


Decomposition

- **Thm**: The edges of $T(G)$ can be decomposed into cliques of size at least three such that each vertex is in exactly $m - n + 1$ cliques.
  - Can break apart graph into pieces that are completely connected, where each vertex is in same number of pieces.
  - Can be used to predict number of edges in $T(G)$.

Decomposition

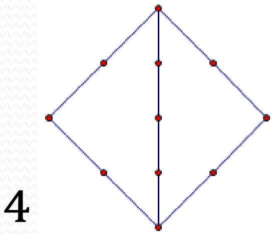




$m = 5$
$n = 4$
$m - n + 1 = 2$

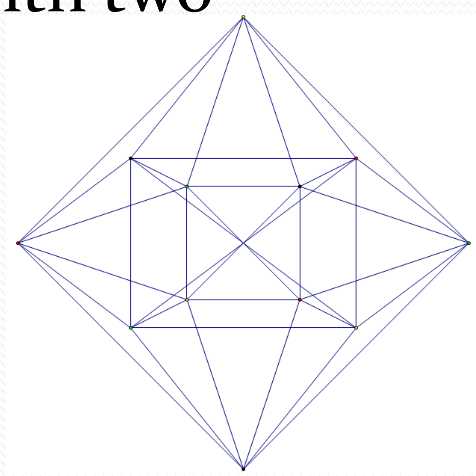
Additional Families

$P_{3,4}$



- Let $P_{n,k}$ be the graph where two vertices are joined by n disjoint paths of edge length k.
- **Thm**: $T(P_{n,k})$ is $(n-1)(2k-1)$-regular.
- **Conj**: $T(P_{n,k})$ is integral (with easily-understood eigenvalues) and vertex transitive.
- $T(P_{n,k})$ could be a new infinite family (with two parameters) of regular integral graphs.
  - These are *really* nice graphs

$T(P_{3,2})$



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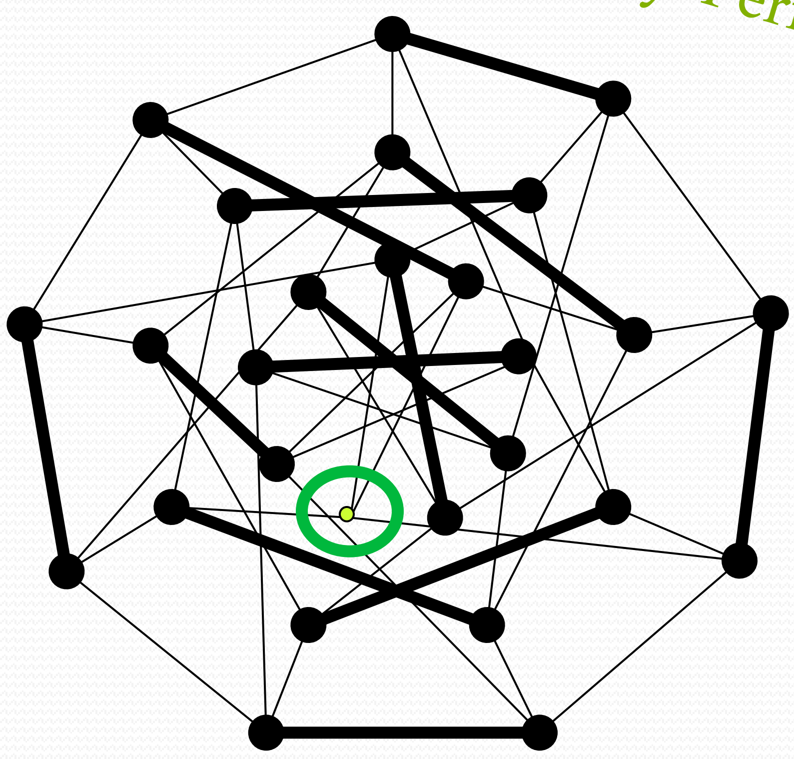
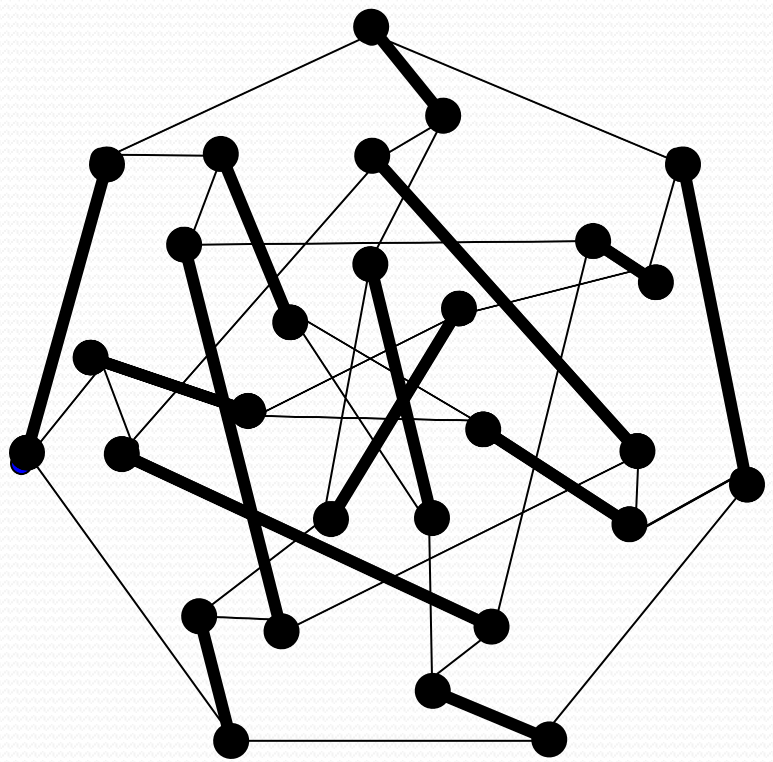
Def. A perfect matching is a set of disjoint edges that covers all of the vertices in a graph.

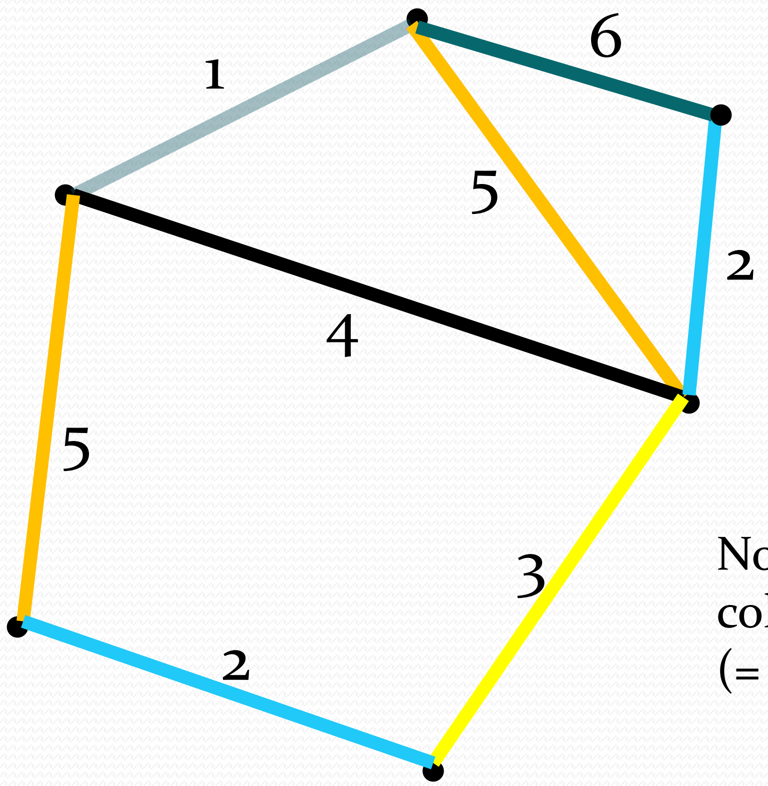


Doyle Graph



Coxeter Graph

Coloring the edges of a graph

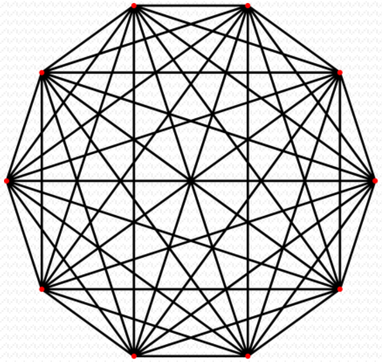


A coloring is an assignment of colors (numbers) to the edges of a graph

A proper coloring has distinct colors at each vertex.

Notice that the color classes for a proper coloring must be disjoint sets of edges (= matchings!)

1-factorizations of K_{2n}



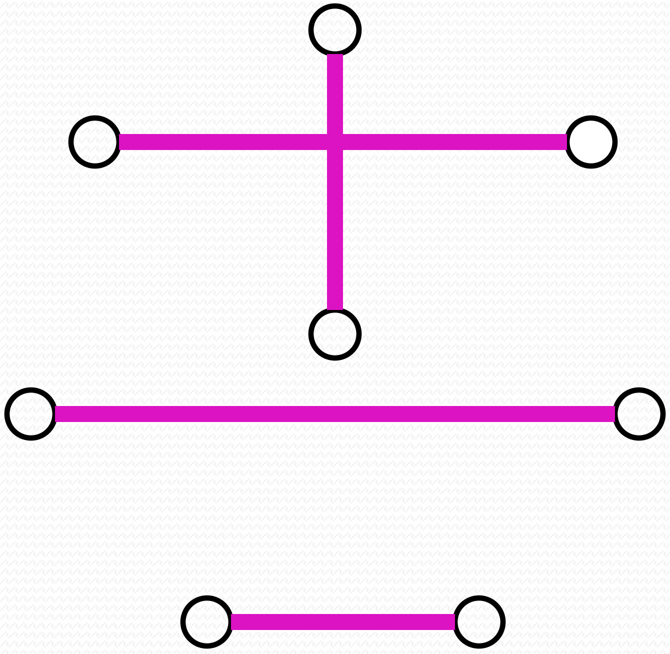
- Lots of not-so-nice ones…

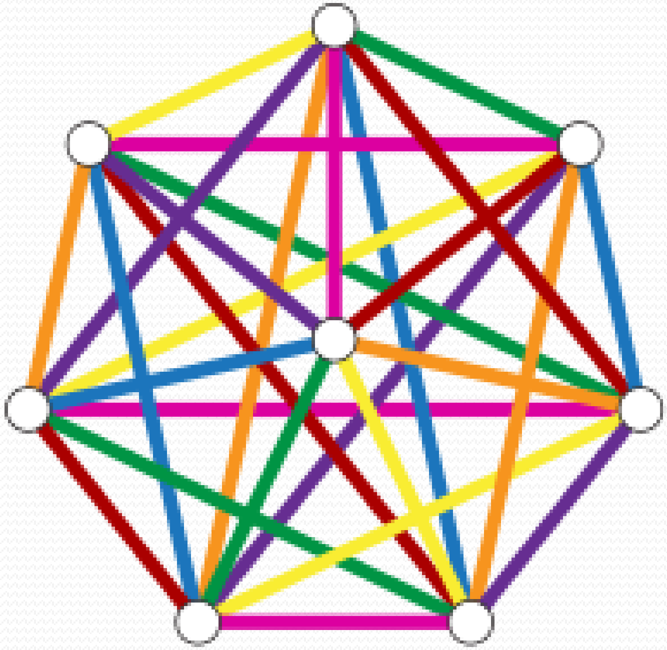
  In fact, of the 396 different rainbow colorings of K_{10}, most look 'random'

- Some very nice ones…

  The most commonly known rainbow coloring of K_{2n} is called GK_{2n}

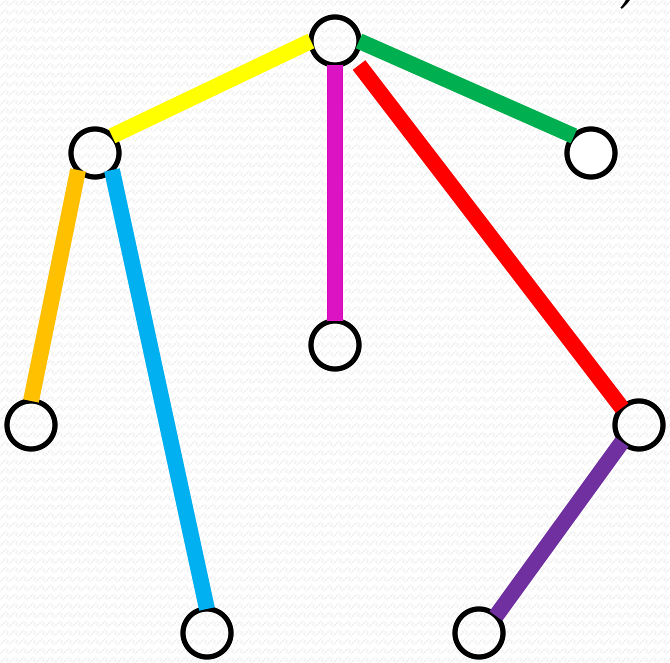
The GK_{2n} 1-factorization

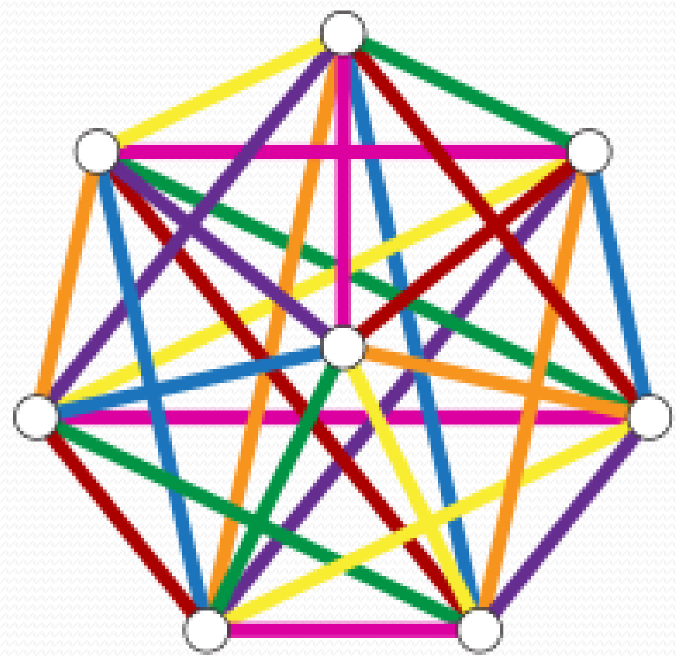




Orthogonal spanning trees

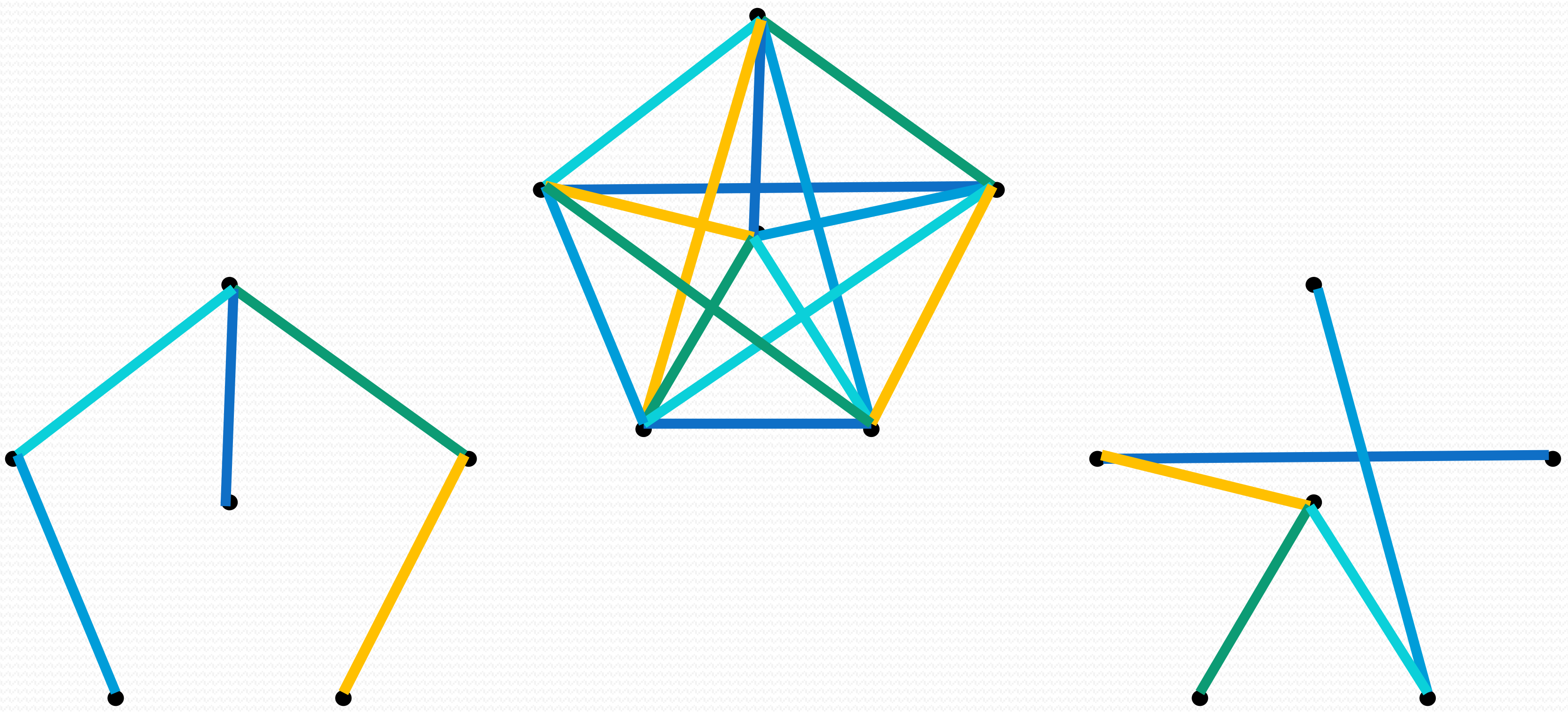
For any 1-factorization of K_{2n}, an orthogonal spanning tree has no 2 edges of the same color!
$(2n-1$ different colors)





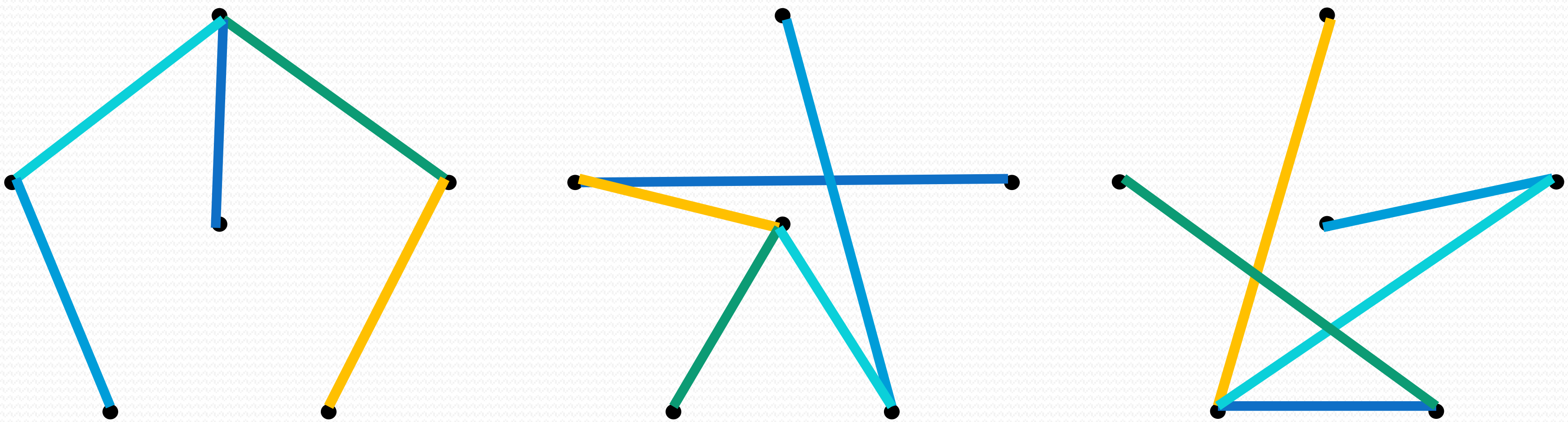
Brualdi-Hollingsworth Theorem

Thm. (1996) Any 1-factorization of K_{2n} has at least 2 disjoint orthogonal spanning trees.



Brualdi-Hollingsworth Conjecture

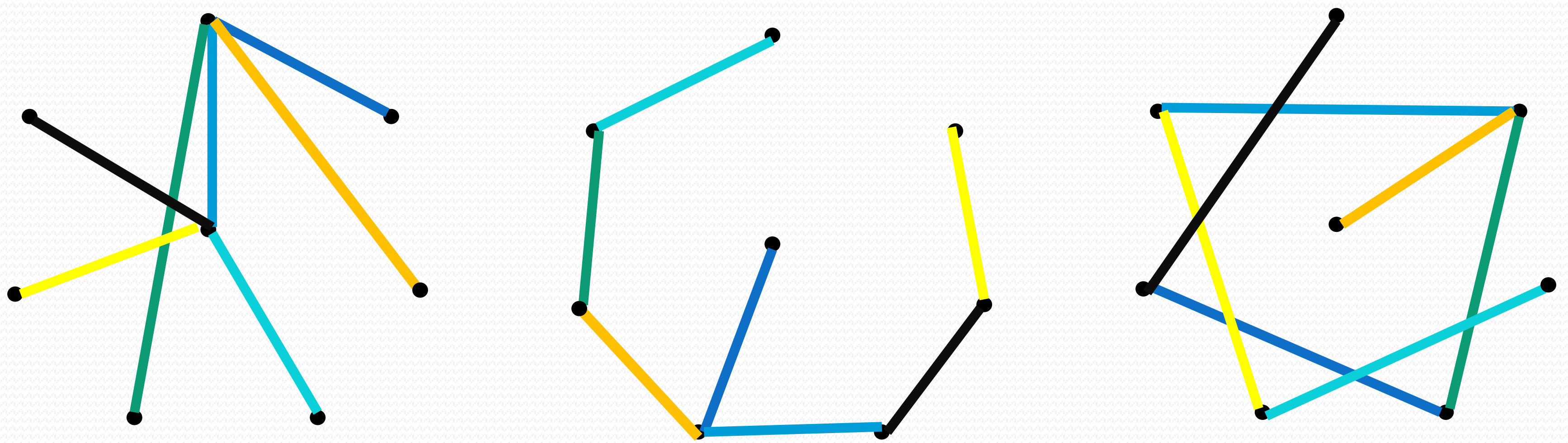
Conj. (1996) Any 1-factorization of K_{2n} has a full set of n disjoint orthogonal spanning trees!



A first step

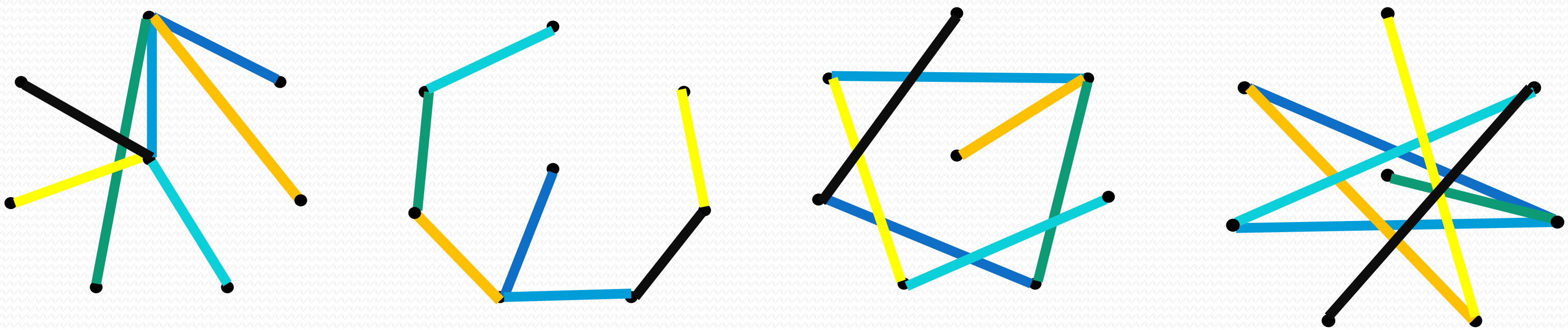
Thm. (Krussel, Marshall, and Verall, 2000)

Any 1-factorization of K_{2n}, has at least 3 disjoint orthogonal spanning trees!



Another step

Thm. (KMV, 2000) If $2n-1$ is a prime of the form $8m+7$ then GK_{2n} has a full set of n disjoint orthogonal spanning trees.



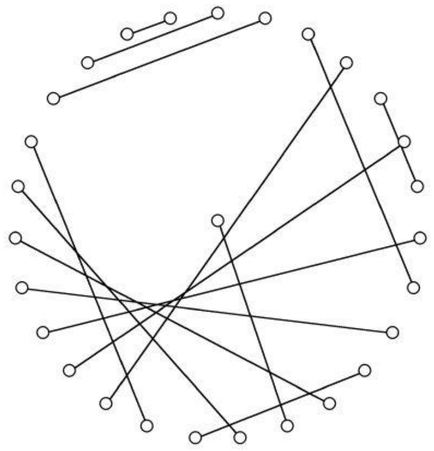
An idea to build upon

- Since GK_{2n} is so nice, the symmetry should help us build nice trees, too.

- Specifically, the colorings rotate around a single vertex. So perhaps the trees should, too.

Rotational 1-factorizations

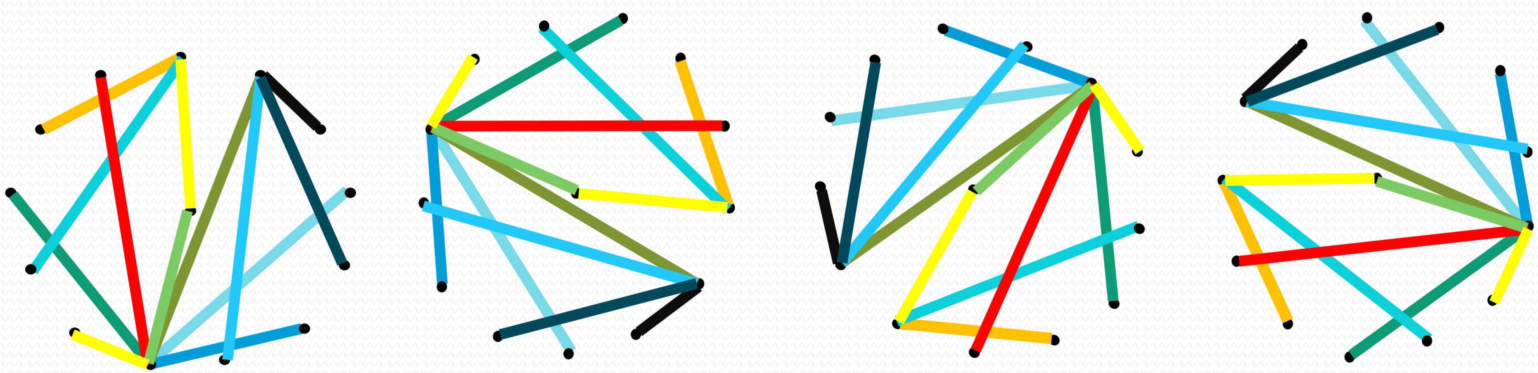
Def. In a rotational 1-factorization, each M_i, can be obtained from M_1 by rotation.



WK_{28}

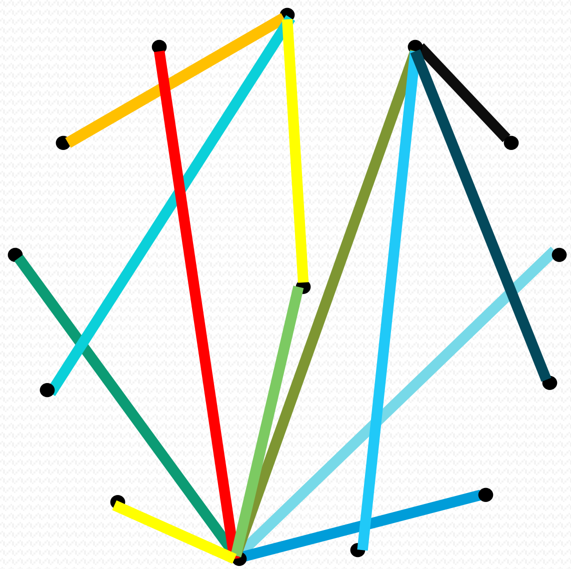
Rotational spanning trees

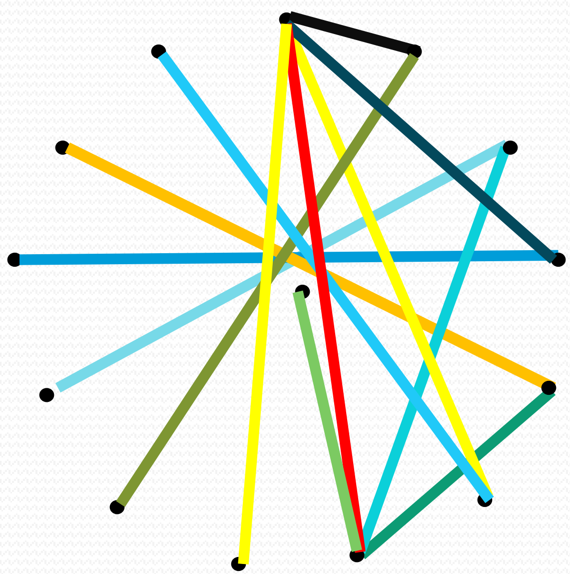
Def. In a rotational set of spanning trees all (but one) of the trees T_i, can be obtained from T_1 by rotation.



Proof of concept

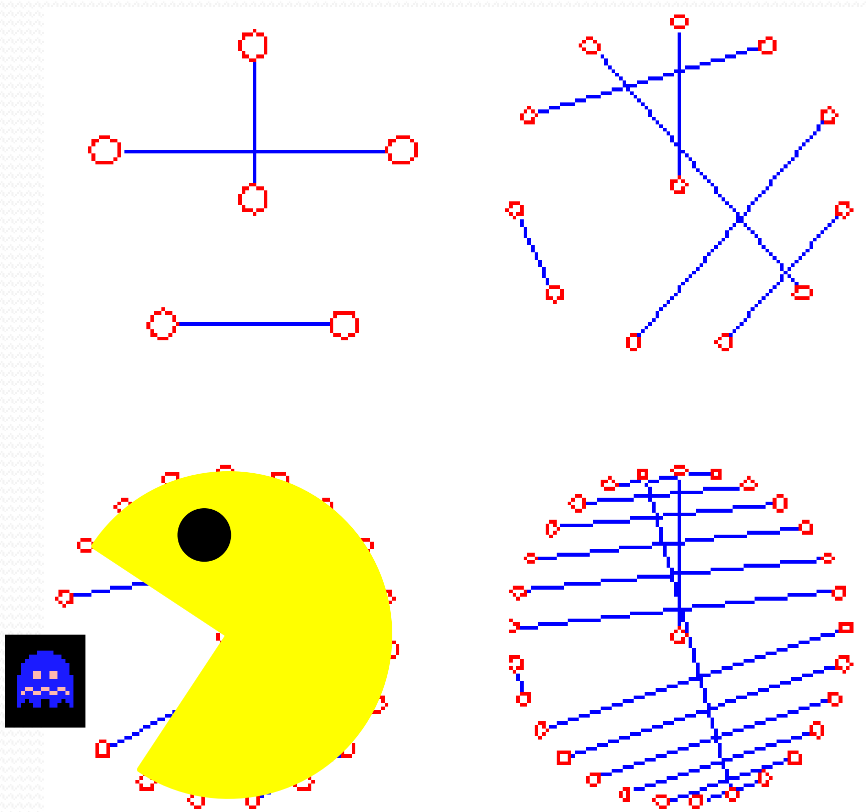
Thm. (Caughman, Krussel) For every n, GK_{2n} has a full rotational set of n disjoint orthogonal spanning trees.





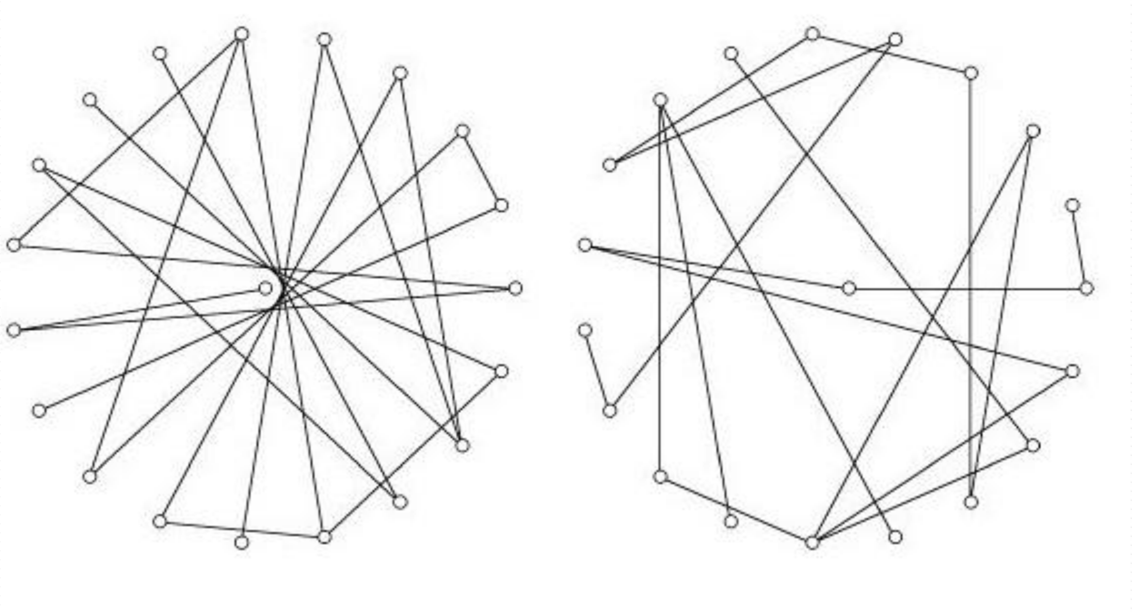
New 1-Factorization

- Called the *half family*, HK_{2n}



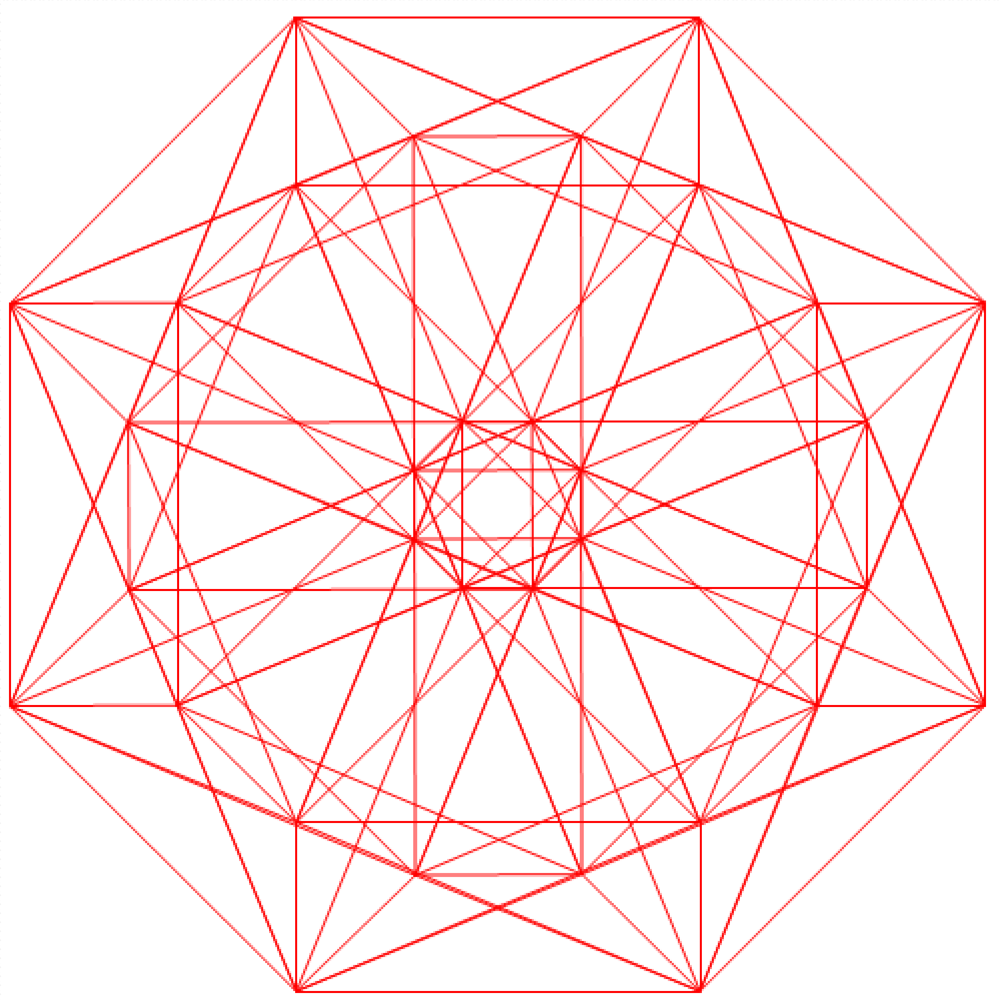
Proposed Extension

Conj. Every rotational 1-factorization of K_{2n} has a full rotational set of orthogonal spanning trees.



Thanks!

- Any questions?



$T(P_{4,2})$